Random analytic functions via random measures

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Prelude: how to look at Brownian motions

Different definitions of Browian motion:

1. Random walks/Lévy's construction;
2. Gaussian process;
3. Axiomatic characterization;
4. Semi-group and heat equation;
5. ...but also random Fourier series.

Given an infinite sequence of i.i.d. standard Gaussians N_k,

$$B(s) = \sum_{k \geq 1} \frac{\sqrt{2}}{\pi} \frac{N_k}{k} \sin(k\pi s), \quad s \in [0,1]$$

defines a Brownian bridge

$$\mathbb{E}[B(s)B(t)] = s(1-t).$$

Random Gaussian field on the unit circle

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Consider $\mathbb{T} := \{e^{i\theta}\}_{\theta\in[0,2\pi)} \in \mathbb{C}$.
We define the random Fourier series (with mean zero on $\mathbb{T}$)

$$X_c(\theta) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (A_n \cos(n\theta) + B_n \sin(n\theta)),$$

where A_n, B_n are i.i.d. $\sim \mathcal{N}(0,1)$.

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It defines (e.g. Sobolev $W^{-s,2}$ for $s > 0$) a Gaussian field X_c on $\mathbb{T}$ with

$$\mathbb{E}[X_c(\theta)X_c(\theta')] = -\log|e^{i\theta} - e^{i\theta'}|$$

$$\left(= \sum_{n=1}^{\infty} \frac{1}{n}\cos(n(\theta-\theta'))\right)$$

This is called "log-correlated field on $\mathbb{T}$".

We define a exponential martingale, formally $: e^{\gamma X_c} :$ (the $: \;,\; :$ is renormalization).

- If $X = B_1 \sin(\theta)$, we define $: e^{\gamma X} : (\theta) = e^{\gamma B_1 \sin(\theta)} e^{-\frac{\gamma^2}{2} \sin^2(\theta)}$.
- If $X = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} (A_n \cos(n\theta) + B_n \sin(n\theta))$, we define for $A \subset \mathbb{T}$,

$$: e^{\gamma X} : (A) = \int_A \prod_{n=1}^{N} e^{\frac{\gamma}{\sqrt{n}} A_n \cos(n\theta)} e^{-\frac{\gamma^2}{2n} \cos^2(n\theta)} \times e^{\frac{\gamma}{\sqrt{n}} B_n \sin(n\theta)} e^{-\frac{\gamma^2}{2n} \sin^2(n\theta)} d\theta.$$

The above is a positive martingale in N.

To recap, starting from the random Fourier series with independent components

$$X_c(\theta) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (A_n \cos(n\theta) + B_n \sin(n\theta)),$$

the product of independent renormalized exponentials is a positive martingale.

The theory of Gaussian multiplicative chaos (started with Kahane) defines the $N \to \infty$ limit $: e^{\gamma X_c} :$ as a random measure on $\mathbb{T}$ denoted μ_γ when $\gamma \in (0, 2)$.

We want to study how each Fourier mode affects the law of μ_c.

We perturbe the Fourier coeffcients (e.g. on B_1). We write

$$X_c = B_1 \sin(\theta) + \left(A_1 \cos(\theta) + \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}(A_n \cos(n\theta) + B_n \sin(n\theta))\right) = B_1 \sin(\theta) + \widetilde{X}_c.$$

Therefore, if $\widetilde{\mu}_c =: e^{\gamma \widetilde{X}_c}$:, we have $\mu_c = e^{B_1 \sin(\theta)} e^{-\frac{1}{2}\sin^2(\theta)} \times \widetilde{\mu}_c$.

Application to negative moments of signed Gaussian multiplicative measure

Decomposition of the Gaussian field:

$$X_c = B_1 \sin(\theta) + \widetilde{X}_c.$$

Decomposition of the random measure:

$$d\mu_c = e^{B_1 \sin(\theta)} e^{-\frac{1}{2}\sin^2(\theta)} \times d\widetilde{\mu}_c.$$

With $F(x) = x^{-p}$ and $p > 0$, we consider

$$\mathbb{E}\left[\left(\int_{\mathbb{T}} \sin(\theta)\, d\mu_\gamma(\theta)\right)^{-p}\right].$$

With ρ parametrizing the random Fourier coefficient B_1, we study

$$v(\rho) := \int_{\mathbb{T}} \sin(\theta) e^{\rho \sin(\theta)} e^{-\frac{1}{2}\sin^2(\theta)} d\widetilde{\mu}_\gamma(\theta).$$

We want to study

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{\rho^2}{2}} |v(\rho)|^{-p} d\rho$$

where

$$v(\rho) := \int_{\mathbb{T}} \sin(\theta) e^{\rho \sin(\theta)} e^{-\frac{1}{2}\sin^2(\theta)} d\widetilde{\mu}_\gamma(\theta).$$

Key: the perturbation is of constant sign

$$\frac{v(\rho)}{d\rho} = \int_{\mathbb{T}} \sin^2(\theta) e^{\rho \sin(\theta)} e^{-\frac{1}{2}\sin^2(\theta)} d\widetilde{\mu}_\gamma(\theta)$$
$$\geq c \inf \left\{ \widetilde{\mu}_\gamma \left[\frac{\pi}{4}, \frac{3\pi}{4} \right], \widetilde{\mu}_\gamma \left[\frac{5\pi}{4}, \frac{7\pi}{4} \right] \right\} > 0.$$

with uniform lower bound given by the independent measure $\widetilde{\mu}_\gamma$.

Existence of small moments

We are reduced to

$$c \int_{\mathbb{R}} e^{-\frac{\rho^2}{2}} |v(\rho)|^{-p} d\rho$$

with $v'(\rho) \geq m > 0$ with some random m depending on $\widetilde{\mu}_\gamma$.

Therefore $v(\rho)$ stays near the singularity for ($\widetilde{\mu}_\gamma$ dependent) time $\sim \frac{1}{m}$.

Upshot of the story: the negative moment of the signed random measure

$$\int_{\mathbb{T}} \sin(\theta) d\mu_\gamma(\theta)$$

is controlled by some negative moment of a positive random measure

$$\inf\left\{\widetilde{\mu}_\gamma\left[\frac{\pi}{4}, \frac{3\pi}{4}\right], \widetilde{\mu}_\gamma\left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]\right\}.$$

This is crucial for the following probabilistic harmonic analysis result:

Theorem (H.-Saksman, 2023+)

The random analytic function φ on the unit disk $\mathbb{D}$, defined by Clark-transforming the random Gaussian multiplicative chaos measure μ on the unit circle $\mathbb{T}$ is almost surely a Blaschke product.

A glimpse of the proof

The proof combines the following ingredients:

1. Properties of the Clark transform (≃Stieltjes transform in e.g. random matrix);
2. Basic theory of bounded analytic functions on the unit disk and Hardy spaces (e.g. Nevanlinna theory and canonical decomposition theorem);
3. A probabilistic version of Frostman's lemma on conformal perturbations;
4. Convexity inequalities from the theory of Gaussian multiplicative chaos etc.

We next studied the zeroes of the random analytic function φ.

Theorem (H.-Saksman, 2023+)

Denote by $\{z_k\}_{k\geq 1} \in \mathbb{D}$ the zeroes of the random analytic function φ on the unit disk $\mathbb{D}$ (defined as Clark-transform of μ_γ). Then almost surely,

$$\sum_{k\geq 1}(1-|z_k|)^\beta < \infty, \quad (\textit{resp.} = \infty),$$

if $1-\frac{\gamma^2}{8} < \beta$ (resp. $1-\frac{\gamma^2}{8} > \beta$).

Accordingly, when γ is larger (more randomness), there are less zeros of φ.

Heuristics on the second result

On a heuristic level, one can guess the threshold $1 - \frac{\gamma^2}{8}$ by using

1. Large deviation and level-set estimates of the Gaussian multiplicative chaos (which level set is responsible for the appearance of zeros of φ);
2. Multifractal analysis of the log-correlated Gaussian field X or the Gaussian multiplicative chaos measure μ_γ (≃Hausdorff dimension of the level set);
3. A probabilistic version of the Riesz theorem on Hilbert transform (that it suffices to study the real part of some random analytic function).

Indeed, the threshold is the Hausdorff dimension of the so-called $\frac{\gamma}{2}$-thick points, determined by Kahane-Peyrière then Hu-Miller-Peres for the Gaussian free field, by Rhodes-Vargas then Bertacco for the Gaussian multiplicative chaos.

Discussions on the second result

The proof (we have several different proofs but this is shortest) includes:

1. A rank-two perturbation strategy $X = V_1 f_1(\theta) + V_2 f_2(\theta) + \widetilde{X}$ (which generalizes the rank-one perturbation method) above;
2. Elementary convex geometry and operator theory to produce the correct directions to perform perturbation ($\simeq$ choosing the basis (f_1, f_2));
3. Harnack inequality to derive a Hilbert transform estimate à la Riesz theorem;
4. Mapping the density of zeroes to level sets of Gaussian multiplicative chaos and use Rhodes-Vargas or Bertacco's result.

History of the problem

Finally, it should be mentioned that research in this direction was suggested by Peltoratksi and Hedenmalm to Saksman about a decade ago.